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Refraction and reflection of a wave when the interface and the media are moving at relativistic speeds

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Explicit expressions for the cosines of the refracted and reflected angles are given as functions of the cosine of the incident angle when the two media (usually fluids) and the interface are moving with relativistic velocities. An application can be the refraction of electromagnetic waves in rarefied but very large clouds of gas moving at relativistic speeds in the expanding universe. [S1063-651X(98)12102-5]

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I. INTRODUCTION

The laws of refraction and reflection of a nonrelativistic wave, when the interface and the media are moving with speeds much smaller than the speed c of light, have been given in a recent paper [1]. We show here that the expressions for reflection are rigorous, while the third step of refraction is approximated to first order. We give here an improved classical expression approximated to second order. We then extend the classical treatment to the case when the speeds are relativistic.

The main application regards the refraction, and sometimes the total internal reflection, of electromagnetic waves in rarefied but very large clouds of gas moving at relativistic speeds in the expanding universe.

A case of interest could be the gravitational lens effect, in which a quasar receding with a speed $\simeq 0.9c$ (where c is the speed of light *in vacuo*) emits light that is deviated by a cluster of galaxies receding with $v \simeq 0.7c$. Actually, two or more beams of light coming from the same quasar can reach the Earth, each beam following a different path, for instance one passing from one part, and the other on the opposite part with respect to the deviating cluster of galaxies. The observed quasar appears, therefore, as split in two or more sources of light. The angular splitting is a few seconds of an arc and therefore a non-negligible contribution can come from the refraction of light through the progressively more rarefied gas surrounding each galaxy and a cluster of galaxies.

II. REFRACTION

We obtain refraction by the Huygens construction, i.e., by the envelope of the refracted (or reflected) waves. In order to perform this construction, the equiphase surfaces (or wave fronts) in the first medium have to be perpendicular to their velocities. This occurs only in the reference system S_0 at rest with the first medium. To have neglected this fact has led Fahy [2] into error (corrected by Cavalleri *et al.* [3]).

We find the cosine $\cos\theta_2$ of the refracted angle in three steps. In the first one, we pass from the laboratory system S to the system S_0 at rest with the first medium (usually a fluid). In the second step, we perform the Huygens construction in S_0 assuming medium 2 at rest with medium 1. In the third step, we consider the actual velocity $\mathbf{u}_2$ of medium 2, and we pass again to the laboratory system S.

A clarification is needed for the interface σ that can be a generic surface with a regular motion (i.e., without discontinuities). Locally, to find the refraction of a narrow wave beam (or ray), we can consider it as a plane (a small portion of the tangent plane). The two limitations (regular motion and narrow beam) imply that the two points A and D of Fig. 2 have infinitesimal differences of velocity. For example, if σ is rotating with the center of rotation between A and D, the velocities of both A and D are infinitesimal, i.e., σ is considered *locally* at rest. For a large beam, we divide it into narrow beams and calculate separately the refraction for each of them, with their *local* velocity $\mathbf{V}$ for σ.

A. First step

We choose a system S of Cartesian axes with the x axis parallel to the velocity $\mathbf{u}_1$ of medium 1 (through which the incoming wave is propagating before refraction). Let $\mathbf{c}_1$ be the wave velocity in S and $\hat{\mathbf{n}}$ be the unit vector perpendicular to the mobile interface σ and directed from medium 1 to medium 2 (see Fig. 1). The incident angle θ_1 in S is given by

$$\cos\theta_1 = \hat{\mathbf{n}}\cdot\mathbf{c}_1/c_1 = \hat{\mathbf{n}}\cdot\hat{\mathbf{c}}_1 . \tag{1}$$

Notice that $\hat{\mathbf{n}}$ is *not* the transformed unit vector of $\hat{\mathbf{n}}_\sigma$ perpendicular to the interface σ in the system S_σ at rest with σ. Simply, $\hat{\mathbf{n}}$ is the unit vector perpendicular to σ as seen by S that should know the plane tangent to the interface (in the small considered region of incidence). To characterize the local interface σ we choose three nearby points $\mathbf{r}_K$, $\mathbf{r}_N$, and $\mathbf{r}_P$, as shown in Fig. 1, such that (for simplicity taking $y_N = y_K$ and $x_P = x_K$)

$$\mathbf{r}_N - \mathbf{r}_K = (x_N - x_K)\hat{\mathbf{e}}_x + (z_N - z_K)\ \hat{\mathbf{e}}_z , \tag{2}$$

$$\mathbf{r}_P - \mathbf{r}_K = (y_P - y_K)\hat{\mathbf{e}}_y + (z_P - z_K)\ \hat{\mathbf{e}}_z , \tag{3}$$

where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ denote the unit vectors of the Cartesian axes. Denoting

$$A = |\mathbf{r}_N - \mathbf{r}_K| = [(x_N - x_K)^2 + (z_N - z_K)^2]^{1/2}, \tag{4}$$

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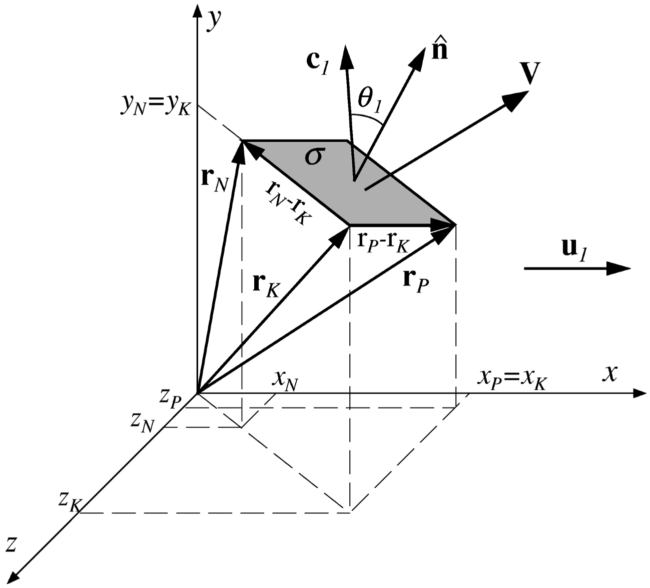

FIG. 1. The local reference system S has been chosen so that the x axis will be parallel to the local velocity $\mathbf{u}_1$ of the first medium. The vectors $\mathbf{r}_K$, $\mathbf{r}_N$, and $\mathbf{r}_P$ denote the positions of three nearby points belonging to the interface σ having local velocity $\mathbf{V}$. The unit vector $\hat{\mathbf{n}}$ is perpendicular to the local element of σ (characterized by $\mathbf{r}_K$, $\mathbf{r}_N$, and $\mathbf{r}_P$). The local velocity of the wave in medium 1 is denoted by $\mathbf{c}_1$ and forms the angle θ_1 with the local normal $\hat{\mathbf{n}}$.

$$B=|\mathbf{r}_P-\mathbf{r}_K|=[(y_P-y_K)^2+(z_P-z_K)^2]^{1/2}, \tag{5}$$

it is

$$\hat{\mathbf{n}}=(\mathbf{r}_N-\mathbf{r}_K)\times(\mathbf{r}_P-\mathbf{r}_K)/AB, \tag{6}$$

hence

$$\hat{n}_x=-(y_P-y_K)(z_N-z_K)/AB,$$

$$\hat{n}_y=-(x_N-x_K)(z_P-z_K)/AB, \tag{7}$$

$$\hat{n}_z=(x_N-x_K)(y_P-y_K)/AB.$$

In order to pass from the laboratory reference system S to the system S_0 at rest with fluid 1, we must specify the kind of clock synchronization we use, since to each kind there follows a corresponding transformation [4]. For instance, if we use the internal synchronization for both S and S_0 (obtained either by Einstein's method or by slow clock transport) the corresponding relativistic transformations are those of Lorentz. If we use the external synchronization, the corresponding relativistic transformations are those of Tangherlini [4]. The latter ones are used in the Appendix as an unusual exercise, while in the main text of this paper we use the Lorentz transformations, which are more familiar. We take the reference system S_0 with the axes parallel to those of reference S and with the x_0 axis of S_0 superimposed and sliding on the x axis of S (the x and x_0 axes are therefore parallel to the velocity $\mathbf{u}_1$ of fluid 1 as observed by S). Consequently, denoting by the subscript 0 the quantities measured in S_0, we have

$$y_0=y,\quad z_0=z,$$

$$x_0=\gamma_1(x-u_1t), \tag{8}$$

$$t_0=\gamma_1(t-xu_1/c^2),$$

where c is the speed of light in vacuum and

$$\gamma_1=(1-u_1^2/c^2)^{-1/2} \tag{9}$$

is the usual relativistic factor.

Because of the longitudinal relativistic contractions, the local interface (where the narrow wave beam impinges) observed by S_0 and denoted by σ_0 is bent differently than the σ observed by S. To use the Huygens construction in the system S_0, the unit vector $\hat{\mathbf{n}}_0$ (measured in S_0) perpendicular to (the plane tangent to) the interface σ_0 must be simultaneous for S_0 so that it is not the transform of $\hat{\mathbf{n}}$. Consequently, $\hat{\mathbf{n}}_0$ can be expressed by the same three nearby points, used in S to characterize the local interface σ, now measured in S_0 and denoted by $\mathbf{r}_{0K}$, $\mathbf{r}_{0N}$, and $\mathbf{r}_{0P}$.

Taking into account that $\mathbf{r}_P$, $\mathbf{r}_K$, $\mathbf{r}_{0P}$, and $\mathbf{r}_{0K}$ are simultaneous since they lie on a plane perpendicular to the x axis, we obtain from Eq. (8)

$$\mathbf{r}_{0P}-\mathbf{r}_{0K}=\mathbf{r}_P-\mathbf{r}_K, \tag{10}$$

so that B_0 corresponding to B defined by Eq. (5) is still equal to B.

On the contrary, $\mathbf{r}_N$ and $\mathbf{r}_{0N}$ are not simultaneous so that $\mathbf{r}_{0N}(t_{0N})-\mathbf{r}_{0K}(t_{0K})$ is *not* the transform of $\mathbf{r}_N(t_N)-\mathbf{r}_K(t_K)$ since $\mathbf{r}_{0N}-\mathbf{r}_{0K}$ and $\mathbf{r}_N-\mathbf{r}_K$ are relevant to the *different* space-time events.

We take $\mathbf{r}_{0K}$, $\mathbf{r}_{0N}$, and $\mathbf{r}_K$ at $t=t_0=0$ and denote by $\mathbf{r}_N^*-\mathbf{r}_K$ the element relevant to the same space-time events of $\mathbf{r}_{0N}-\mathbf{r}_{0K}$ but measured by the laboratory observer S (while $\mathbf{r}_{0N}-\mathbf{r}_{0K}$ is measured by S_0 at rest with medium 1). Their components transversal to $\mathbf{u}_1$ are the same, while their longitudinal components are related to each other, as derivable from $x^*=\gamma_1\,(x_0+u_1\,t_0)$ with $t_{0N}=t_{0K}=0$, by

$$x_N^*-x_K=\gamma_1(x_{0N}-x_{0K}). \tag{11}$$

For S, there is a time interval $t_N^*-t_K$, for the events judged as simultaneous by S_0, derivable from $0=t_0=\gamma_1\,(t^*-u_1x^*/c^2)$, which is

$$t_N^*-t_K=(x_N^*-x_K)u_1/c^2. \tag{12}$$

During this time interval $\mathbf{r}_N$ moves with velocity $\mathbf{V}$ of the local interface reaching $\mathbf{r}_N^*$ so that

$$\mathbf{r}_N^*-\mathbf{r}_K=\mathbf{r}_N-\mathbf{r}_K+\mathbf{V}(x_N^*-x_K)u_1/c^2. \tag{13}$$

Projecting Eq. (13) on $\mathbf{u}_1$, i.e., on the x axis, gives

$$x_N^*-x_K=x_N-x_K+V_x(x_N^*-x_K)u_1/c^2, \tag{14}$$

from which we get

$$x_N^*-x_K=(x_N-x_K)(1-V_xu_1/c^2)^{-1}. \tag{15}$$

Then we obtain by Eqs. (11) and (15)

$$x_{0N}-x_{0K}=\gamma_1^{-1}(x_N-x_K)(1-V_xu_1/c^2)^{-1}. \tag{16}$$

Projecting Eq. (13) on the y and z axes, using Eq. (15), and taking into account that, because of Eq. (8), it is $y_N^* = y_{0N}$ and $z_N^* = z_{0N}$, gives, respectively,

$$y_N^* - y_K = y_{0N} - y_{0K} = V_y(x_N - x_K)(1 - V_x u_1/c^2)^{-1} u_1/c^2, \tag{17}$$

since $y_N = y_K$ (see Fig. 1), and

$$z_N^* - z_K = z_{0N} - z_{0K} = z_N - z_K + V_z(x_N - x_K) \times (1 - V_x u_1/c^2)^{-1} u_1/c^2. \tag{18}$$

Finally, we obtain by Eqs. (16), (17), and (18),

$$\mathbf{r}_{0N} - \mathbf{r}_{0K} = \hat{\mathbf{e}}_x \frac{\gamma_1^{-1}(x_N - x_K)}{(1 - V_x u_1/c^2)} + \hat{\mathbf{e}}_y \left[\frac{V_y(x_N - x_K)}{(1 - V_x u_1/c^2)} \frac{u_1}{c^2} \right] + \hat{\mathbf{e}}_z \left[z_N - z_K + \frac{V_z(x_N - x_K)}{(1 - V_x u_1/c^2)} \frac{u_1}{c^2} \right], \tag{19}$$

so that A_0, corresponding to A defined by Eq. (4), turns out to be given by

$$A_0 = |\mathbf{r}_{0N} - \mathbf{r}_{0K}| = \left\{ \frac{(x_N - x_K)^2[\gamma_1^{-2} + (V_y^2 + V_z^2)u_1^2/c^4]}{(1 - V_x u_1/c^2)^2} + (z_N - z_K)^2 + \frac{2\,(z_N - z_K) V_z u_1\,(x_N - x_K)}{c^2(1 - V_x u_1/c^2)} \right\}^{1/2}. \tag{20}$$

We now have all the elements to define the unit vector $\hat{\mathbf{n}}_0$ measured in S_0 by an expression similar to Eq. (6),

$$\hat{\mathbf{n}}_0 = (\mathbf{r}_{0N} - \mathbf{r}_{0K}) \times (\mathbf{r}_{0P} - \mathbf{r}_{0K})/A_0 B, \tag{21}$$

where $\mathbf{r}_{0N} - \mathbf{r}_{0K}$ is given by Eq. (19), $\mathbf{r}_{0P} - \mathbf{r}_{0K}$ by Eq. (10), B by Eq. (5), and A_0 by Eq. (20).

The wave velocity $\mathbf{c}_{01}$ in S_0 is given, since $\mathbf{u}_1 = u_1\,\hat{\mathbf{e}}_x$, by

$$\mathbf{c}_{01} = (1 - \mathbf{u}_1 \cdot \mathbf{c}_1/c^2)^{-1}[(c_{1x} - u_1)\hat{\mathbf{e}}_x + \gamma_1^{-1}(c_{1y}\hat{\mathbf{e}}_y + c_{1z}\hat{\mathbf{e}}_z)]. \tag{22}$$

The incident angle θ_{01} in S_0 is obtained from

$$\cos\theta_{01} = \hat{\mathbf{n}}_0 \cdot \mathbf{c}_{01}/c_{01} = \hat{\mathbf{n}}_0 \cdot \hat{\mathbf{c}}_{01}, \tag{23}$$

with $\hat{\mathbf{n}}_0$ expressed by Eq. (21).

The three mutually nearby points $\mathbf{r}_K$, $\mathbf{r}_N$, $\mathbf{r}_P$ are chosen so that $\mathbf{r}_{0K}$, $\mathbf{r}_{0N}$, $\mathbf{r}_{0P}$ and their order in the vector product (21) bring about $\cos\theta_{01} > 0$.

B. Second step

The Huygens construction is performed in S_0 considering the fluid in the second medium as being at rest with the first one. The situation of two fluids at relative rest in spite of the fact that their boundary plane moves, is theoretical and useful as an intermediary step to find the final solution. However, such a situation could practically be performed by a thin, porous piston in a cylinder (filled with a fluid) which keeps two different pressures and densities in the two parts of the closed cylinder just by moving. The piston can also be substituted by a shock wave of pressure.

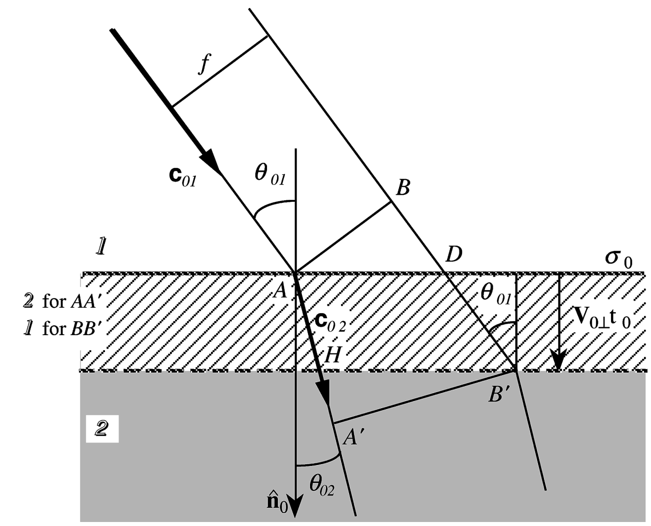

FIG. 2. A wave has velocity $\mathbf{c}_{01}$ in medium 1 and equiphase surface AB perpendicular to $\mathbf{c}_{01}$ if the observer S_0 is at rest with medium 1. An interface having velocity $\mathbf{V}_0$ separates medium 1 from medium 2. $\hat{\mathbf{n}}_0$ is the unit vector perpendicular (for S_0) to the interface and directed from 1 to 2. When a wave ray impinges on the interface at A the wave is refracted in medium 2 (considered at rest with S_0) with velocity $\mathbf{c}_{02}$. Point B of the wave front reaches the moving interface in B' while point A reaches A' in medium 2 at rest with S_0 so that the equiphase surface $A'B'$ is still perpendicular to the refracted ray AHA'. This is the Huygens construction for media at rest but moving interface, $A'B'$ being the envelope of the spherical waves radiated by the points of the interface successively reached by the impinging wave front. In the general case of media moving with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, we add relativistically $\mathbf{u}_1$ to $\mathbf{c}_{01}$ and $\mathbf{u}_2$ to $\mathbf{c}_{02}$.

Let $\mathbf{V}$ be the velocity of the interface in the laboratory system S. The corresponding velocity in S_0 is

$$\mathbf{V}_0 = (1 - \mathbf{u}_1 \cdot \mathbf{V}/c^2)^{-1}[(V_x - u_1)\hat{\mathbf{e}}_x + \gamma_1^{-1}(V_y\hat{\mathbf{e}}_y + V_z\hat{\mathbf{e}}_z)]. \tag{24}$$

Both V and V_0 can, in general, be comparable with c.

What is effective is the component

$$V_{0\perp} = \mathbf{V}_0 \cdot \hat{\mathbf{n}}_0 = \beta c. \tag{25}$$

The unit vector $\hat{\mathbf{n}}_0$ is drawn so that $\mathbf{c}_{01} \cdot \hat{\mathbf{n}}_0 > 0$. Media 1 and 2 contain the incident and refracted wave, respectively. If σ_0 were at rest, there would be no ambiguity about which one is the incident wave. However, if $V_{0\perp} > c_{01}\cos\theta_{01}$, it is the interface σ_0 that reaches the fleeing wave and we have to exchange medium 1 for 2 in Fig. 2. Consequently, if σ_0 is at rest, medium 1 is always the one *not* containing $\hat{\mathbf{n}}_0$ (drawn starting from the interface). If σ_0 is in motion, medium 1 is that not containing $\hat{\mathbf{n}}$ only if

$$s = \mathrm{sgn}\,(c_{01}\cos\theta_{01} - V_{0\perp}) \tag{26}$$

is positive, medium 1 is the one containing $\hat{\mathbf{n}}_0$ if s is negative.

We consider the first case, i.e., $s=+$, in Fig. 2 where the Huygens construction is plotted with respect to observer S_0. The treatment of this second step is the same as in the nonrelativistic case since for S_0 there is no composition of velocities and an observer can describe any motion.

With reference to Fig. 2, AB is the trace of the equiphase front in medium 1 at rest (i.e., observed in system S_0) so that it is perpendicular to the velocity $\mathbf{c}_{01}$. The wave ray which impinges on the boundary plane at A begins to travel in medium 2 with the velocity $\mathbf{c}_{02}$ along AA'. At time t_0, when the phase front reaches A', the same phase started in B at time $t_0=0$ reaches the moving boundary plane in B'. Since the ray section BB' has always been in medium 1 and the ray section AA' in medium 2, it is

$$t_0=|AA'|/c_{02}=|BB'|/c_{01}. \tag{27}$$

The refracted phase front $A'B'$ is perpendicular to AA' because in this second step of our solution the medium 2 is still considered at rest with S_0. This phase front is obtained as the envelope of the spherical waves radiated by each point of the boundary plane reached by the incoming wave.

We see from Fig. 2 that $|BB'|=c_{01}\,t_0$ may also be written as

$$c_{01}\,t_0=|BD|+|DB'|=|AD|\,\sin\theta_{01}+V_{0\perp}6t_0/\cos\theta_{01}. \tag{28}$$

Similarly we may write $|AA'|=c_{02}\,t_0$ as

$$c_{02}\,t_0=|A'H|+|HA|=|B'H|\,\sin\theta_{02}+V_{0\perp}t_0/\cos\theta_{02}, \tag{29}$$

where

$$|B'H|=|AD|+V_{0\perp}t_0(\tan\theta_{01}-\tan\theta_{02}). \tag{30}$$

Obtaining t_0 from Eq. (28) and substituting it in Eq. (29), where Eq. (30) is used, gives, after simplifying the factor $|AD|$ that appears in both sides,

$$\begin{aligned} c_{02}\,\sin\theta_{01}=&\sin\theta_{02}\left(c_{01}-\frac{V_{0\perp}}{\cos\theta_{01}}\right)\\ &+V_{0\perp}\,\sin\theta_{01}\,\sin\theta_{02}\left(\frac{\sin\theta_{01}}{\cos\theta_{01}}-\frac{\sin\theta_{02}}{\cos\theta_{02}}\right)\\ &+V_{0\perp}\,\frac{\sin\theta_{01}}{\cos\theta_{02}}. \end{aligned} \tag{31}$$

Simplifying Eq. (31), and calling

$$m=V_{0\perp}\,\sin\theta_{01},\quad p=c_{01}-V_{0\perp}\,\cos\theta_{01},\quad q=c_{02}\,\sin\theta_{01}, \tag{32}$$

we obtain

$$m\,\cos\theta_{02}+p\,\sin\theta_{02}=q, \tag{33}$$

If $V_{0\perp}=-|V_{0\perp}|$, all the preceding expressions keep their validity.

The same Eq. (33) is obtained in the second case ($s=-$), as shown in Ref. [1], since this second step is the same in both the relativistic and nonrelativistic treatments. The only difference is given by the connection (24) between $\mathbf{V}$ and $\mathbf{V}_0$ (which in the nonrelativistic case reduces to $\mathbf{V}_0=\mathbf{V}-\mathbf{u}_1$). The solution of Eq. (33), taking into account all the cases [$s=\pm$, with s given by Eq. (26)] and subcases ($s=-$, $p>0$ and $p<0$ as examined in Ref. [1]), is

$$\cos\theta_{02}=\frac{mq+sp(m^2+p^2-q^2)^{1/2}}{m^2+p^2}, \tag{34}$$

with m, p, and q given by Eq. (32).

Since c_{02} is known (it is the speed of the wave in the medium at rest), projecting the direction of $\mathbf{c}_{02}$ on the normal $\hat{\mathbf{n}}_0$ to the local interface and on the normal to $\hat{\mathbf{n}}_0$ lying in the refraction plane, it is

$$\mathbf{c}_{02}=\hat{\mathbf{n}}_0c_{02}\,\cos\theta_{02}+(\hat{\mathbf{n}}_0\times\mathbf{c}_{01})\times\hat{\mathbf{n}}_0\,|\hat{\mathbf{n}}_0\times\mathbf{c}_{01}|^{-1}c_{02}\,\sin\theta_{02}, \tag{35}$$

where $\cos\theta_{02}$ (hence $\sin\theta_{02}$) is given by Eq. (34), $\hat{\mathbf{n}}_0$ and $\mathbf{c}_{01}$ are given by Eqs. (21) and (22), respectively.

C. Third step

The third step of our procedure is now introduced by considering the motion of the second fluid.

Since we have chosen the x axis parallel to $\mathbf{u}_1$, as said in Sec. II A, the velocity $\mathbf{u}_{02}$ of medium 2 with respect to S_0 is given by

$$\mathbf{u}_{02}=(1-\mathbf{u}_1\cdot\mathbf{u}_2/c^2)^{-1}[(u_{2x}-u_1)\hat{\mathbf{e}}_x+\gamma_1^{-1}(u_{2y}\hat{\mathbf{e}}_y+u_{2z}\hat{\mathbf{e}}_z)], \tag{36}$$

where γ_1 is given by Eq. (9). Notice that even if $u_{02}\ll c$ we must use the relativistic composition (36) since u_1 and u_2 are relativistic.

If $u_{02}\ll c_{02}\ll c$, as usually occurs in the nonrelativistic case studied in Ref. [1], then the velocity $\mathbf{c}_{02}^*$, still measured by S_0 but in the second fluid moving with velocity $\mathbf{u}_{02}$, has been written in Ref. [1] as

$$\mathbf{c}_{02}^*\simeq\mathbf{c}_{02}+\mathbf{u}_{02}\quad\text{if } u_{02}\ll c_{02}\ll c. \tag{37}$$

Notice that $\mathbf{c}_{02}$ is now the velocity of the wave in a third reference system S_2 at rest with the second medium. Obviously, its absolute value c_{02} is equal to that used in Sec. II B, since there medium 2 was considered at rest. On the contrary, $\mathbf{c}_{02}^*$ is measured in S_0 and $c_{02}^*\neq c_{02}$ because of the drag due to the second medium.

If even u_1 and u_2 are much less than c, then $\mathbf{u}_{02}\simeq\mathbf{u}_2-\mathbf{u}_1$, so that Eq. (37) reduces to Eq. (16) of Ref. [1]. As a clarification of what was done in Ref. [1], it appears from Fig. 3 that Eq. (37) is approximate to the first order in u_{02}/c_{02}. Actually, in the prerelativistic case, where there is complete drag of the wave by part of the moving medium, the center of the spherical wave produced by the incident wave in A moves to F given by $AF=\mathbf{u}_{02}t_0$, where $t_0=|BB'|/c_{01}$. Now the envelope of the spherical waves in medium 2 is the wave front $B'G$, tangent in G to the spheri-

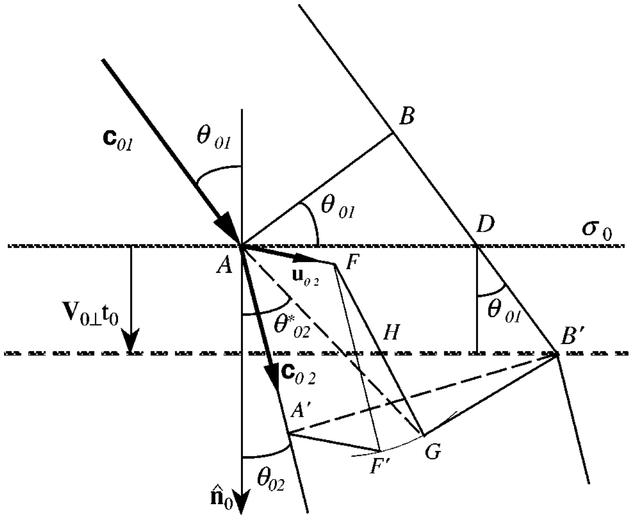

FIG. 3. The center of the wave produced in A is transported to F and the vector AF is given by $\mathbf{u}_{02}\,|BB'|/c_{01}$. The envelope of the wave centered in F and starting from B' is the wave front $B'G$. The rigorous direction in the nonrelativistic case would be AG and not $AF'=(\mathbf{c}_{02}+\mathbf{u}_{02})\,|BB'|/c_{01}$. The result may be expressed as $\mathbf{c}^*_{02}=\mathbf{c}_{02}+\mathbf{u}_{02}+\mathbf{u}_{02}\cdot\hat{\mathbf{c}}_{02}[(\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01})/|\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01}|]\times\hat{\mathbf{c}}_{02}+\mathbf{o}(u_{02}/c_{02})^2$, which coincides with $\mathbf{c}_{02}+\mathbf{u}_{02}$ when $\mathbf{u}_{02}$ is perpendicular to $\mathbf{c}_{02}$.

cal wave centered in F. The simple addition of the velocities would give the point F' (such that $A'F'=\mathbf{u}_{02}t_0$) instead of G, i.e., a direction AF' for the refracted wave instead of AG. We clearly see that $F'G/FG=o(u_{02}/c_{02})$ and that Eq. (37) is rigorous only when $\mathbf{u}_{02}$ is perpendicular to $\mathbf{c}_{02}$.

In the general case, the first order approximation in u_{02}/c_{02} is just of the same order of the ratio between the second and the first term in Eq. (37). The worst approximation occurs when $\hat{\mathbf{u}}_{02}=\hat{\mathbf{c}}_{02}$. An improved expression with respect to Eq. (37) is

$$\mathbf{c}^*_{02}=\mathbf{c}_{02}+\mathbf{u}_{02}+\mathbf{u}_{02}\cdot\hat{\mathbf{c}}_{02}\,\frac{(\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01})}{|\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01}|}\times\hat{\mathbf{c}}_{02}, \tag{38}$$

which is approximated to second order in (u_{02}/c_{02}).

Now, even with an acoustic wave (so that c_{02} is the speed of sound), it is $u_{02}/c_{02}\ll 1$ so that Eq. (38) is widely satisfied. Even if u_1 and u_2 are very large, as is the case for the cloud of gas surrounding a galaxy receding at relativistic speed, the relative velocity $\mathbf{u}_{02}$ can be very small if we divide the moving and shrinking cloud of gas in many layers so that the relative velocity $\mathbf{u}_{02}$ between two adjacent layers is much smaller than c_{02}.

We can proceed in the same way in the relativistic case, as well. In the system S_2 at rest with the second fluid, the wave front AB and the local interface form an angle different from that observed in S_0 because of the longitudinal relativistic contraction. However, this is a second order effect in u_{02}/c and therefore of the same order as Eq. (38).

We therefore assume that the system S_2 measures a velocity for the refracted wave equal to the velocity $\mathbf{c}_{02}$ measured by S_0 when the second medium was assumed at rest with medium 1 and given by Eq. (35).

In order to obtain the velocity $\mathbf{c}^*_{02}$ of the refracted wave as measured by S_0, we must transform $\mathbf{c}_{02}$ measured in the reference system S_2 (at rest with fluid 2 and therefore moving with velocity $\mathbf{u}_{02}$ with respect to S_0) to $\mathbf{c}'_{02}$ measured in the system S_0 (at rest with fluid 1) and then add the first order correction appearing in the classical Eq. (38), i.e.,

$$\mathbf{c}^*_{02}=\mathbf{c}'_{02}+\mathbf{u}_{02}\cdot\hat{\mathbf{c}}_{02}\,\frac{(\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01})}{|\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01}|}\times\hat{\mathbf{c}}_{02}. \tag{39}$$

Since the x axis is in general not parallel to $\mathbf{c}_{02}$, we must start from the Lorentz transformation between S_0 and the system S_2 (at rest with fluid 2) in vector form,

$$\begin{gathered}\mathbf{r}_0=\mathbf{r}_2+\hat{\mathbf{u}}_{02}(\gamma_{02}-1)\hat{\mathbf{u}}_{02}\cdot\mathbf{r}_2+\gamma_{02}\mathbf{u}_{02}t_2,\\ t_0=\gamma_{02}(t_2+\mathbf{u}_{02}\cdot\mathbf{r}_2/c^2),\end{gathered} \tag{40}$$

where

$$\gamma_{02}=(1-u_{02}^2/c^2)^{-1/2}. \tag{41}$$

We derive from Eq. (40), taking into account that we assumed $d\mathbf{r}_2/dt_2=\mathbf{c}_{02}$,

$$\mathbf{c}'_{02}=\frac{d\mathbf{r}_0}{dt_0}=\gamma_{02}^{-1}\,\frac{\mathbf{c}_{02}+\hat{\mathbf{u}}_{02}(\gamma_{02}-1)\hat{\mathbf{u}}_{02}\cdot\mathbf{c}_{02}+\gamma_{02}\mathbf{u}_{02}}{1+\mathbf{u}_{02}\cdot\mathbf{c}_{02}/c^2}. \tag{42}$$

For $u_{02},c_{02}\ll c$, Eq. (42) reduces to Eq. (37). For $u_{02}\ll c$ but c_{02} relativistic, expanding Eqs. (42) to first order gives

$$\mathbf{c}'_{02}\simeq\mathbf{c}_{02}+\mathbf{u}_{02}-\mathbf{c}_{02}\mathbf{u}_{02}\cdot\mathbf{c}_{02}/c^2 \quad \text{if } u_{02}\ll c. \tag{43}$$

If we denote by $n=c/c_{02}$ the refraction index in the second fluid, Eq. (43) becomes

$$\mathbf{c}'_{02}\simeq\mathbf{c}_{02}+u_{02}(\hat{\mathbf{u}}_{02}-\hat{\mathbf{c}}_{02}\;\hat{\mathbf{u}}_{02}\cdot\hat{\mathbf{c}}_{02}/n^2), \tag{44}$$

which expresses Fizeau's drag coefficient.

We now add the corrective term of Eq. (38) to either Eq. (42) or Eq. (43), since Eq. (38) is approximated to within $o(u_{02}/c_{02})^2$. Choosing the simple Eq. (43), we obtain

$$\mathbf{c}^*_{02}\simeq\mathbf{c}_{02}+\mathbf{u}_{02}-\mathbf{c}_{02}\mathbf{u}_{02}\cdot\mathbf{c}_{02}/c^2+\mathbf{u}_{02}\cdot\hat{\mathbf{c}}_{02}\,\frac{(\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01})}{|\hat{\mathbf{n}}_0\times\hat{\mathbf{c}}_{01}|}\times\hat{\mathbf{c}}_{02}, \tag{45}$$

where now, differently from the classical Eq. (38), all the quantities are relativistic, with $\hat{\mathbf{n}}_0$ given by Eq. (21), $\hat{\mathbf{c}}_{01}$ by Eq. (22), $\mathbf{u}_{02}$ by Eq. (36), and $\mathbf{c}_{02}$ by Eq. (35).

To obtain $\mathbf{c}_2$ we transform $\mathbf{c}^*_{02}$ from the system S_0 (at rest with medium 1) to the laboratory system S,

$$\mathbf{c}_2=(1+\mathbf{u}_1\cdot\mathbf{c}^*_{02}/c^2)^{-1}[(c^*_{02x}+u_1)\hat{\mathbf{e}}_x+\gamma_1^{-1}(c^*_{02y}\hat{\mathbf{e}}_y+c^*_{02z}\hat{\mathbf{e}}_z)], \tag{46}$$

where $\mathbf{u}_1$ is the known velocity of fluid 1 with respect to the laboratory observer and $\mathbf{c}^*_{02}$ is given by Eq. (45), with $\mathbf{u}_{02}$ and $\mathbf{c}_{02}$ given by Eqs. (36) and (35), respectively. Finally,

$$\cos\theta_2=\hat{\mathbf{n}}\cdot\mathbf{c}_2/c_2, \tag{47}$$

where $\mathbf{c}_2$ is given by Eq. (46) and $\hat{\mathbf{n}}$ is the local normal to the interface as seen by the laboratory observer S and given by Eq. (6).

III. REFLECTION

Reflection on a mobile mirror may be treated in a similar way with the obvious simplification $\mathbf{u}_1=\mathbf{u}_2=\mathbf{u}$, $c_1=c_2$, $c_{01}=c_{02}=c_0$, and $|AA'|=|BB'|=a=c_0t_0$ (see Fig. 2 of Ref. [1]). The unit vector $\hat{\mathbf{n}}$ perpendicular to the reflecting mirror as judged by the laboratory system S is still given by Eq. (6) with A and B given by Eqs. (4) and (5), respectively. The first step always consists in passing from S to the system S_0 at rest with the fluid and to find the new unit vector $\hat{\mathbf{n}}_0$ perpendicular to the mirror as judged by S_0. This unit vector $\hat{\mathbf{n}}_0$ is expressed by Eq. (21), in which B is given by Eq. (5) and A_0 by Eq. (20). The wave velocity $\mathbf{c}_{01}$ in S_0 is given by Eq. (22), and the incident angle θ_{01} by Eq. (23), i.e., $\cos\theta_{01}=\mathbf{c}_{01}\cdot\hat{\mathbf{n}}_0/c_0$. The component $V_{0\perp}$ (of the mirror velocity $\mathbf{V}_0$) along the unit normal $\hat{\mathbf{n}}_0$ perpendicular to the surface and measured by S_0 is given by Eq. (25), where $\mathbf{V}_0$, expressed by the quantities in S, is given by Eq. (24).

The Huygens construction in S_0, shown in Fig. 3 of Ref. [1], leads to Eqs. (28)–(33). The only difference (with respect to refraction) regards solution (34), since we now have to choose the other sign for the radical that appears in the solutions of the second degree equation in $\cos\theta_{02}$. Since the radical is now a perfect square because $c_{01}=c_{02}=c_0$, the solution is therefore

$$\cos\theta_{02}=\hat{\mathbf{c}}_{02}\cdot\hat{\mathbf{n}}_0=\frac{mq-p(c_0\cos\theta_{01}-\beta c)}{p^2+m^2}. \tag{48}$$

The sign of s in Eq. (34) is now substituted by the term inside the parentheses in Eq. (48).

Once $\cos\theta_{02}$ is obtained, one calculates $\sin\theta_{02}$ and, by Eq. (35), the direction $\mathbf{c}_{02}/c_0$ of the reflected wave ray.

The third step consists in returning to the laboratory system S by means of Eq. (46), where, for reflection, $\mathbf{c}^*_{02}=\mathbf{c}_{02}$. Finally, $\cos\theta_2$ is still given by Eq. (47).

In the case of reflection, the first and the second media are the same, so that the relative velocity $\mathbf{u}_{02}=0$. Consequently, both the classical Eq. (38) and the relativistic Eq. (45) give rigorously $\mathbf{c}^*_{02}=\mathbf{c}_{02}$. The result for reflection is therefore rigorous and not approximated to second order as occurs for refraction.

IV. CONCLUSIONS

We have improved the classical treatment [1] of refraction when the two media and the interface are moving with nonrelativistic velocities. The improved expression is Eq. (38), which is approximated to second order in u_{02}/c_{02}, as the corresponding relativistic expression Eq. (45).

The classical treatment for reflection (that does not need the third step) was already rigorous and we have here given the rigorous relativistic treatment.

The solution for the cosine of the refracted wave is given by Eq. (47), with $\mathbf{c}_2$ given by Eq. (46), where $\mathbf{c}^*_{02}$ is expressed by Eq. (39), with $\mathbf{c}'_{02}$ given by Eq. (42). Then $\mathbf{u}_{02}$ and $\mathbf{c}_{02}$, which appear in Eq. (42), are given by Eqs. (36) and (35), respectively. In turn, $\cos\theta_{02}$ in Eq. (35) is given by Eq. (34), where m, p, and q are given by Eq. (32). Finally, $V_{0\perp}$ is given by Eq. (25) with $\hat{\mathbf{n}}_0$ and $\mathbf{V}_0$ expressed by Eqs. (21) and (24), respectively, both $\mathbf{u}_1$ and $\mathbf{V}$ being known, and $\mathbf{r}_{0N}-\mathbf{r}_{0K}$ and A_0 being given by Eqs. (19) and (20), respectively.

The same expressions hold for reflection, the only difference being that $\cos\theta_{02}$ is given by Eq. (48) instead of by Eq. (34) and that $\mathbf{c}^*_{02}=\mathbf{c}_{02}$, since fluid 2 is the same as fluid 1.

These expressions, as said in the Introduction, may have applications in astrophysics, in particular to find small corrections to the gravitational lens effect because of the relativistic motion of the very rarefied but very large clouds of gas surrounding both the emitting quasar and the deflecting cluster of galaxies.

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APPENDIX

Before the three fundamental papers of Mansouri and Sexl [4], physicists used only the Lorentz transformations, which are a consequence of the internal synchronization of clocks, performed either by the Einstein method or by the slow clocks transport. After their work, it became clear that there are infinite possible relativistic transformations corresponding to infinite possible synchronizations of clocks as occurs in prerelativistic, or Galilean, kinematics. However, while in the latter there is only one very simple transformation (that of Galileo), in special relativity there are two convenient and simple transformations: (i) of Lorentz, and (ii) of Tangherlini. The second one corresponds to the ‘‘external’’ synchronization, in which the second observer S_0 synchronizes his own clocks by local coincidences with the clocks belonging to the first observer S. In this way there is conservation of simultaneity of separate events and the transformations are no longer symmetric. For instance, longitudinal rods at rest with S_0 are measured as contracted by S (as by the Lorentz transformations), but the longitudinal rods at rest with S are measured as lengthened by S_0 (differently from what occurs by the Lorentz transformations). Similarly, the rates of clocks at rest with S_0 are measured as slowed down by S, but the rates of clocks at rest with S are measured as increased by S_0. Moreover, the speed of light is c and isotropic for S only but not for S_0, i.e., it is no longer invariant.

The Tangherlini transformations (corresponding to the external synchronization) are

$$t_0=\gamma_1^{-1}t \tag{A1}$$

and

$$x_0=\gamma_1(x-u_1t),$$

$$y_0=y,\quad z_0=z,$$

or, in vector form,

$$\mathbf{r}_0=\mathbf{r}+(\gamma_1-1)\mathbf{r}\cdot\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1-\gamma_1\mathbf{u}_1 t. \tag{A2}$$

The direct measurements performed by S are the same so that we still have Eqs. (1)–(7). For S_0 we still have Eqs. (21) and (10) but, instead of Eq. (19), we have the much simpler expression

$$\mathbf{r}_{0N}-\mathbf{r}_{0K}=\hat{\mathbf{e}}_x\gamma_1(x_N-x_K)+\hat{\mathbf{e}}_z(z_N-z_K), \tag{A3}$$

since the external synchronization implies the conservation of simultaneity of separate events. Consequently,

$$A_0=|\mathbf{r}_{0N}-\mathbf{r}_{0K}|=[\gamma_1^2\,(x_N-x_K)^2+(z_N-z_K)^2]^{1/2}, \tag{A4}$$

instead of the much more complicated Eq. (20).

The wave velocity $\mathbf{c}_{01}$ in S_0 can be derived from Eq. (A1), and it is

$$c_{01y}=\frac{dy_0}{dt_0}=\frac{dy}{\gamma_1^{-1}dt}=\gamma_1 c_{1y},\quad c_{01z}=\gamma_1 c_{1z},$$

$$c_{01x}=\frac{dx_0}{dt_0}=\frac{\gamma_1(dx-u_1dt)}{\gamma_1^{-1}dt}=\gamma_1^2(c_{1x}-u_1),$$

or, in vector form,

$$\mathbf{c}_{01}=\gamma_1[\gamma_1(c_{1x}-u_1)\hat{\mathbf{e}}_x+c_{1y}\hat{\mathbf{e}}_y+c_{1z}\hat{\mathbf{e}}_z], \tag{A5}$$

instead of Eq. (22).

Similarly, the velocity $\mathbf{V}_0$ (measured by S_0 using the Tangherlini transformations) of the moving interface σ_0 is

$$\mathbf{V}_0=\gamma_1[\gamma_1(V_x-u_1)\hat{\mathbf{e}}_x+V_y\hat{\mathbf{e}}_y+V_z\hat{\mathbf{e}}_z], \tag{A6}$$

and the velocity $\mathbf{u}_{02}$ of medium 2 (measured by S_0) is expressed by

$$\mathbf{u}_{02}=\gamma_1[\gamma_1\,(u_{2x}-u_1)\hat{\mathbf{e}}_x+u_{2y}\hat{\mathbf{e}}_y+u_{2z}\hat{\mathbf{e}}_z]. \tag{A7}$$

The velocity $\mathbf{c}_{02}^*$ (still measured by S_0) of the wave in the moving medium 2 is still given by Eq. (39). To obtain $\mathbf{c}_{02}'$ we must find the transformations from medium 2 to medium 1, taking into account that the Tangherlini transformations are *not* symmetric.

If we maintain the same directions for the Cartesian axes (i.e., $\hat{\mathbf{e}}_1=\hat{\mathbf{u}}_1$), we have that $\mathbf{u}_2$ is *not* parallel to the x axis, and the inverse transformations from the reference system S_2 (at rest with fluid 2) to the laboratory system S are

$$t=\gamma_2 t_2,$$

$$\mathbf{r}=\mathbf{r}_2+\hat{\mathbf{u}}_2(\gamma_2^{-1}-1)\mathbf{r}_2\cdot\hat{\mathbf{u}}_2+\gamma_2\mathbf{u}_2 t_2, \tag{A8}$$

where

$$\gamma_2=(1-u_2^2/c^2)^{-1/2}. \tag{A9}$$

Notice that $\gamma_2\neq\gamma_{02}$ given by Eq. (41).

Substituting Eqs. (A8) into Eqs. (A1) and (A2) gives

$$t_0=\gamma_1^{-1}\gamma_2 t_2,$$

$$\begin{aligned}\mathbf{r}_0=\mathbf{r}_2&+\hat{\mathbf{u}}_2(\gamma_2^{-1}-1)\mathbf{r}_2\cdot\mathbf{u}_2+\gamma_2\mathbf{u}_2 t_2\\&+(\gamma_1-1)\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1\cdot[\mathbf{r}_2+\hat{\mathbf{u}}_2(\gamma_2^{-1}-1)\mathbf{r}_2\cdot\hat{\mathbf{u}}_2+\gamma_2\mathbf{u}_2 t_2]\\&-\gamma_1\gamma_2\mathbf{u}_1 t_2,\end{aligned} \tag{A10}$$

from which we get

$$\begin{aligned}\mathbf{c}_{02}'=\frac{d\mathbf{r}_0}{dt_0}=\gamma_1\gamma_2^{-1}\{&\mathbf{c}_{02}+\hat{\mathbf{u}}_2(\gamma_2^{-1}-1)\mathbf{c}_{02}\cdot\hat{\mathbf{u}}_2+\gamma_2\mathbf{u}_2\\&+(\gamma_1-1)\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1\cdot[\mathbf{c}_{02}+\hat{\mathbf{u}}_2(\gamma_2^{-1}-1)\\&\times\mathbf{c}_{02}\cdot\hat{\mathbf{u}}_2+\gamma_2\mathbf{u}_2]-\gamma_1\gamma_2\mathbf{u}_1\},\end{aligned} \tag{A11}$$

where $\mathbf{c}_{02}$ is still given by Eq. (35).

Finally, in order to transform $\mathbf{c}_{02}^*$ from S_0 to the laboratory system S, we use the transformation inverse to that given by Eq. (A2), i.e.,

$$\mathbf{r}=\mathbf{r}_0+(\gamma_1-1)\mathbf{r}_0\cdot\hat{\mathbf{u}}_1\hat{\mathbf{u}}_1+\gamma_1\mathbf{u}_1 t_0, \tag{A12}$$

from which we get

$$\mathbf{c}_2=\frac{d\mathbf{r}}{dt}=\gamma_1^{-1}[\mathbf{c}_{02}^*+(\gamma_1^{-1}-1)\mathbf{c}_{02}^*\cdot\hat{\mathbf{u}}_1+\gamma_1\mathbf{u}_1]. \tag{A13}$$

The cosine of the refracted angle is still given by Eq. (47) (of the main text), where $\mathbf{c}_2$ is now given by Eq. (A13), where $\mathbf{u}_1$ is known, and $\mathbf{c}_{02}^*$ is still given by Eq. (39) with $\mathbf{c}_{02}'$ given by Eq. (A11), in which $\mathbf{c}_{02}$ is still given by Eq. (35) with $\cos\vartheta_{02}$ given by Eqs. (34) and (32). Finally, $V_{0\perp}$ is given by Eq. (25) with $\hat{\mathbf{n}}_0$ and $\mathbf{V}_0$ expressed by Eqs. (21) and (A6), respectively, both $\mathbf{u}_1$ and $\mathbf{V}$ being known, and $\mathbf{r}_{0N}-\mathbf{r}_{0K}$ and A_0 being given by Eqs. (A3) and (A4), respectively.

Concluding, the use of the Tangherlini transformations leads to a much simpler expression for $\hat{\mathbf{n}}_0$ and to a somewhat more complicated expression for $\mathbf{c}_{02}'$ than that of Lorentz.

[1] A. Ascoli, C. Bernasconi, and G. Cavalleri, Phys. Rev. E **54**, 6291 (1996).

[2] E. F. Fahy, Nature (London) **188**, 396 (1960).

[3] G. Cavalleri, L. Galgani, G. Spavieri, and G. Spinelli, Lett. Nuovo Cimento **17**, 25 (1976).

[4] For the other possible relativistic transformations consequent to different clock synchronizations, see R. Mansouri and R. V. Sexl, Gen. Relativ. Gravit. **8**, 497 (1977); **8**, 515 (1977); **8**, 809 (1977); M. Podlaha, Indian J. Theor. Phys. **26**, 189 (1978); T. S. Sjödin, Nuovo Cimento B **51**, 229 (1979); G. Cavalleri and G. Spinelli, Phys. Found. **13**, 122 (1983); G. Cavalleri and C. Bernasconi, Nuovo Cimento B **104**, 545 (1989); C. M. Will, Phys. Rev. D **45**, 403 (1992); G. Cavalleri and E. Tonni, Ch. J. Syst. Engin. Electr. **6**, 147 (1995).